Measuring X-ray Polarization: An Introduction

Steven Ehlert
IXPE Project Scientist
IXPE/NICER Workshop

What do I want to convey to you all in this talk?

1. Measuring angles with the IXPE detector
2. The modulation factor μ
3. The minimum detectable polarization MDP
4. Stokes parameters I, Q, and U
5. Significance tests using Stokes parameters and χ^2 testing
6. Using weights to improve sensitivity

My main goal is help you understand the results produced by software that is designed to perform these calculations.

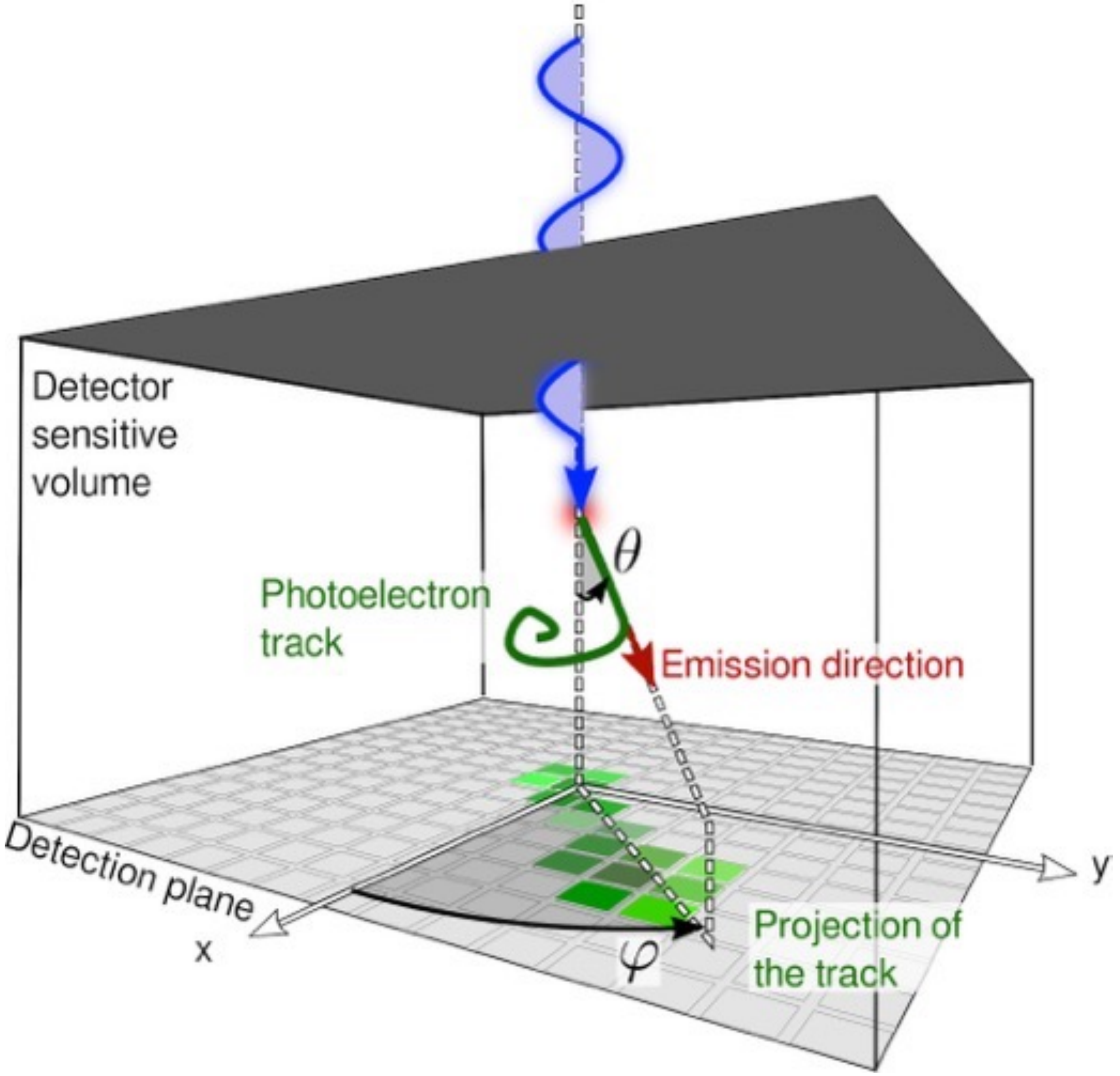
Please don't hesitate to read over these charts off-line!

Some References

I am going to go through the fundamental statistics of X-ray polarization using the conventions of Muleri 2022.

Other references such as Weisskopf et al 2010, Kislat et al 2015, Montgomery and Swank 2015, and Strohmayer 2017 use slightly different conventions that lead to identical results.

The IXPE detectors measure photoelectron tracks.

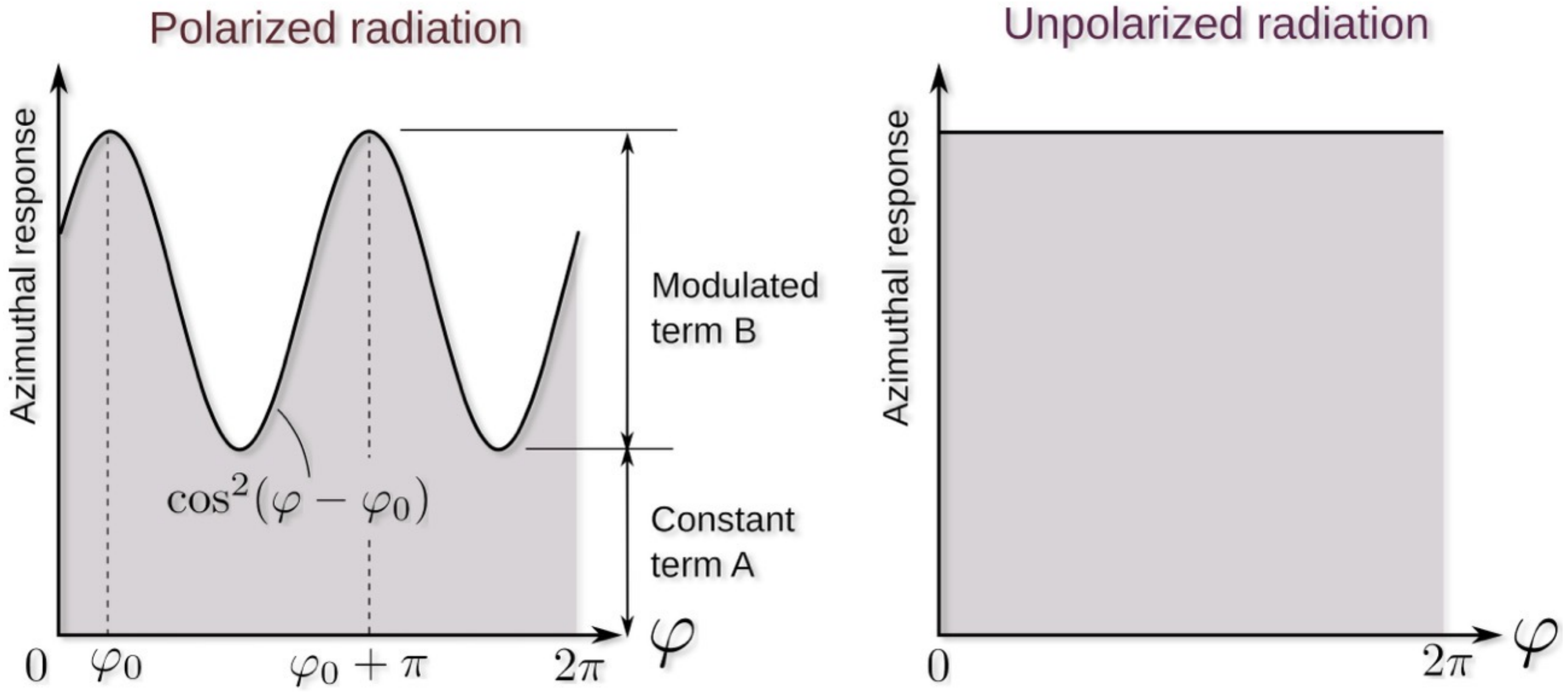


The crucial angle for measuring polarization is φ, which is points back (statistically) to the electric field vector of the incoming X-ray.

$$\frac{d\sigma}{d\Omega} \propto \cos^2(\varphi - \varphi_0)\ \sin^2\theta$$

A useful number to keep in mind for these tracks in DME: the average work per photoelectric ion pair is about 28 eV, meaning we get about 35 electrons per keV of X-ray photon.

We construct a distribution of track angles to measure polarization.

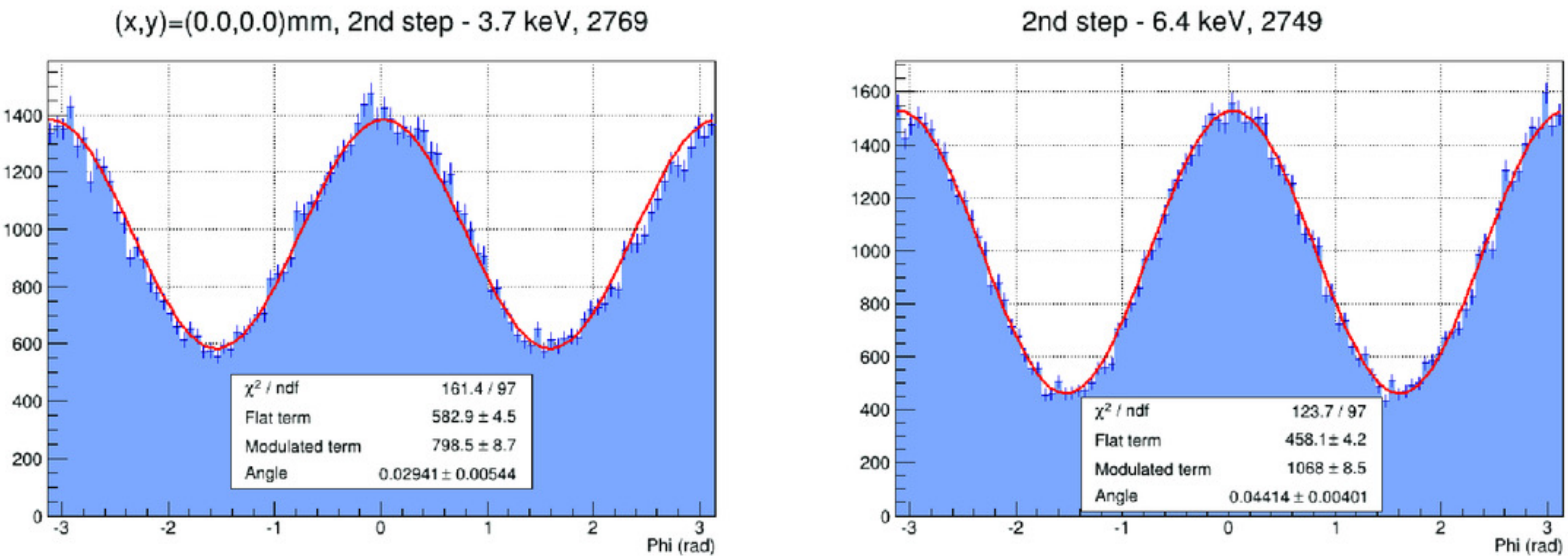


$$\mathcal{M}(\varphi) = A + B\ \cos^2(\varphi - \varphi_0)$$

No detector reconstructs angles perfectly, however.

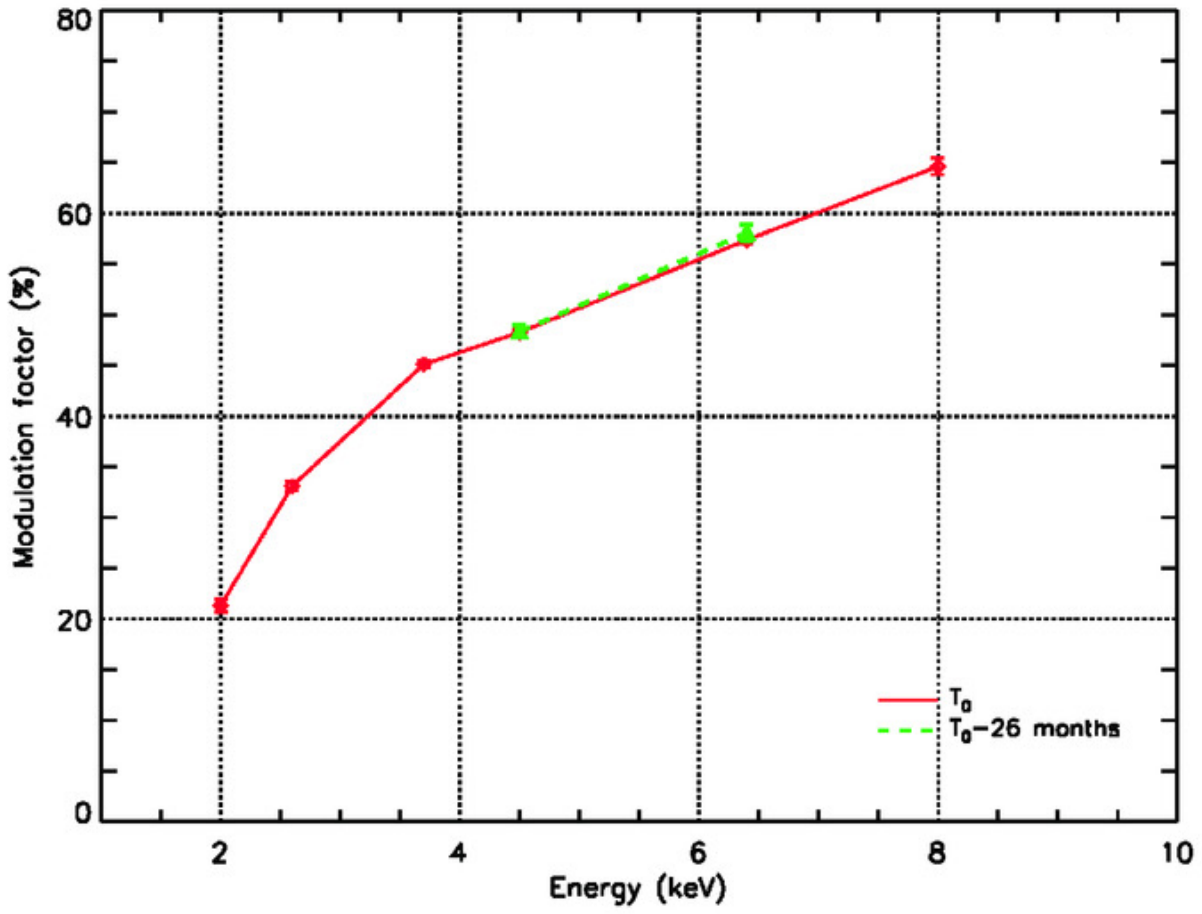
The modulation factor μ converts between the measured distribution parameters and the physical polarization degree.

$\Pi_X = \frac{1}{\mu}\frac{B}{B+2A}$ with μ restricted to the range of 0-1.



The modulation factor is a detector property.

The modulation factor ultimately depends on the detectors ability to reconstruct the initial track direction for each event – events where this is easy/precise have higher values of μ than events where uncertainties are larger. The CALDB includes the modulation factor as a function of energy.



Minimum detectable polarization is the crucial number.

When we measure our distribution of track angles and find a modulated component, how do we know it is not just a statistical fluctuation?

We can calculate the probability density function of detecting a particular measurement of Π_X with N events assuming that the TRUE polarization is $\Pi_X = 0$ (Weisskopf et al 2010)

$$P(\Pi_X \,|\Pi_{X,true} = 0)d\Pi_X = \frac{N}{2}\,\mu^2\,\Pi_X e^{-\frac{N\,\mu^2\Pi_X^2}{4}} d\Pi_X$$

Minimum detectable polarization is the crucial number.

When we measure our distribution of track angles and find a modulated component, how do we know it is not just a statistical fluctuation?

We can calculate the probability density function of detecting a particular measurement of Π_X with N events assuming that the TRUE polarization is $\Pi_X = 0$ (Weisskopf et al 2010)

$$P(\Pi_X \,|\Pi_{X,true} = 0)d\Pi_X = \frac{N}{2}\,\mu^2\,\Pi_X e^{-\frac{N\,\mu^2\Pi_X^2}{4}} d\Pi_X$$

We can integrate this PDF to get a cumulative distribution function, and solve for the value of MDP corresponding to a confidence level C

$$C = 1 - \ e^{\frac{-MDP^2\ N\ \mu^2}{4}}$$

Minimum detectable polarization is the crucial number.

Typically we set C = 0.99 (99% confidence interval) and solve for MDP to be

$$MDP_{99} = \frac{4.29}{\mu\sqrt{N}}$$ or with background $$MDP_{99} = \frac{4.29\sqrt{N_{src} + N_{bg}}}{\mu \, N_{src}}$$

What this MDP means is that in cases where the TRUE polarization is zero and you ran many experiments where you gather N photons, 99% of them would have MEASURED polarization degrees less than MDP_{99} .

Note that you need a LOT of counts to measure low polarization degrees – for a 2% MDP we need more almost 300,000 counts assuming $\mu = 0.4$.

IXPE Imaging X-Ray Polarimetry Explorer

Stokes Parameters

For IXPE and other polarimeters, it is usually more convenient to work in Stokes parameters. For individual event k, we construct its Stokes parameters as

$$i_k = 1$$
$$q_k = 2\cos 2\varphi_k$$
$$u_k = 2\sin 2\varphi_k$$

And then construct I, Q, and U as the sums over all relevant events. The polarization degree and angle are determined as

$$\Pi_X = \frac{1}{\mu\, I}\sqrt{Q^2 + U^2}$$
$$\varphi_X = \frac{1}{2}\tan^{-1}\frac{U}{Q}$$

Why Stokes Parameters?

- The Stokes Parameters Q and U are independent of one another, unlike polarization degree and angle.
- With sufficiently large numbers of counts, the central limit theorem ensures Q and U are normally distributed variables with equal standard deviation (i.e. $\sigma_Q = \sigma_U = \sigma$).
- Stokes parameters are additive – if you have background events you can simply subtract the background Stokes parameters from your src + background Stokes parameters.
- In terms of the Stokes parameter standard deviation σ, the uncertainties for Π_X and φ_X are $\sigma_\Pi = \sigma$ and $\sigma_\varphi = \frac{\sigma}{2\Pi_X}$. Note that in cases where there is a non-detection the polarization angle is unconstrained.

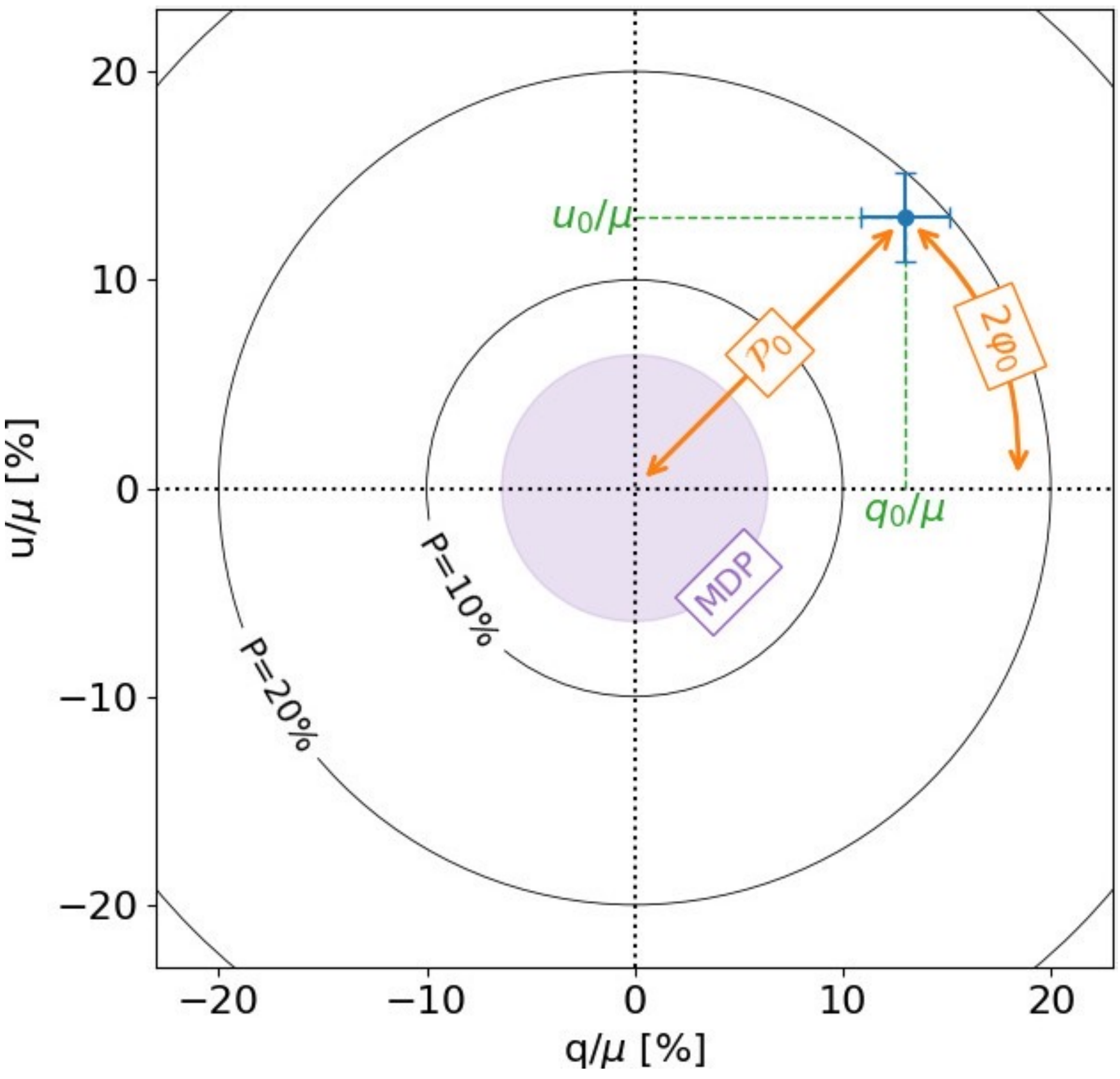
Stokes parameters offer an intuitive way to test for detection.

We can test for detection significance using a null hypothesis and the Stokes parameters with their uncertainties (reminder – they are equal for Q and U).

Assuming that the true values of Q and U are zero, our normalized distance from the origin

$$\chi^2 = \frac{Q^2 + U^2}{\sigma^2},$$

is distributed as a χ^2 distribution with 2 degrees of freedom. This can immediately be used to determine the significance of your detection. Gives the exact same formula for MDP as above.

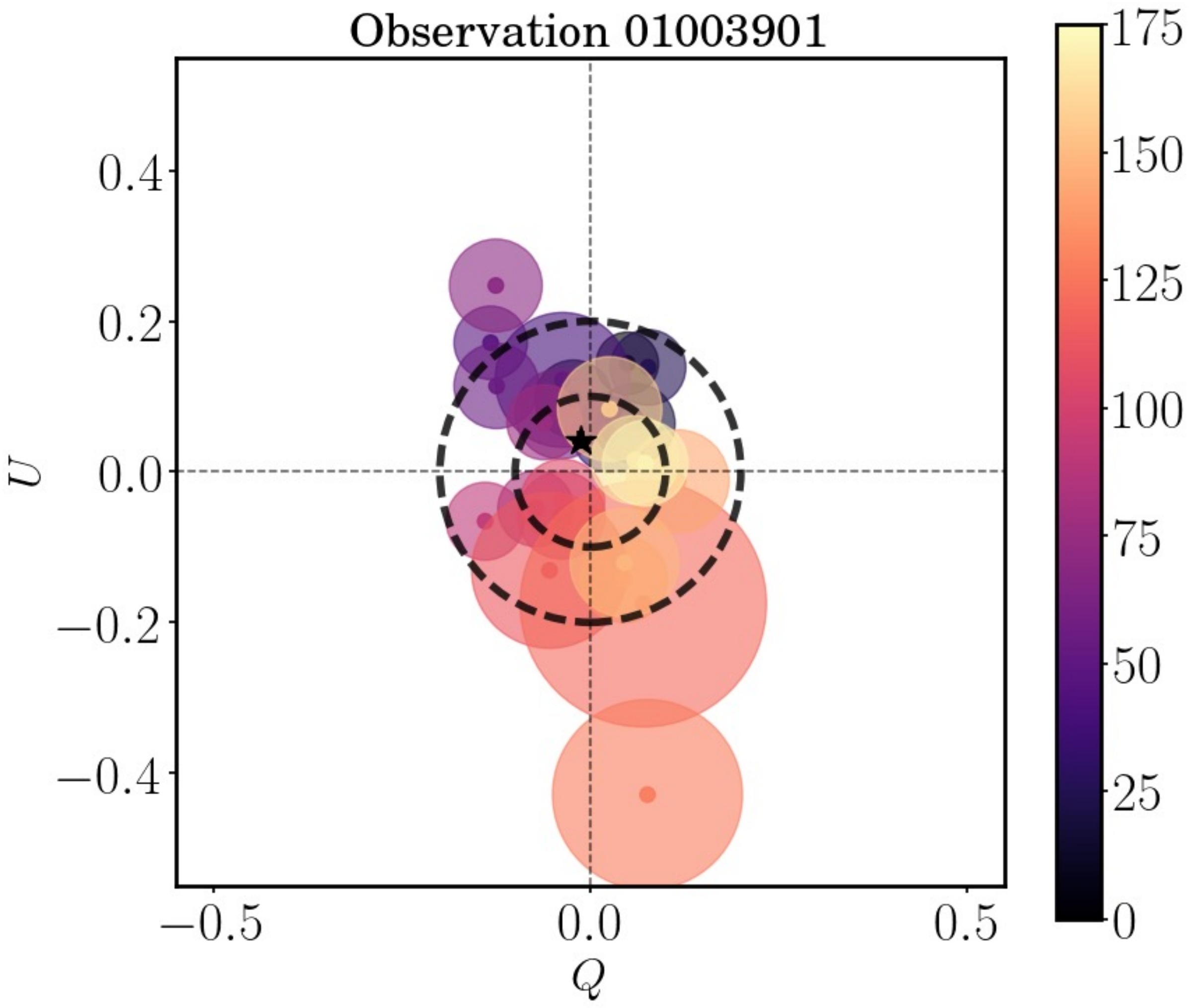


Further Tests using Stokes Parameters

Tests for detection significance can be done in any time/energy/spatial bin you wish – albeit with lower signal to noise.

Similar tests using χ^2 can be used for more sophisticated hypothesis testing – for example, are the Stokes parameters consistent with being constant as a function of time?

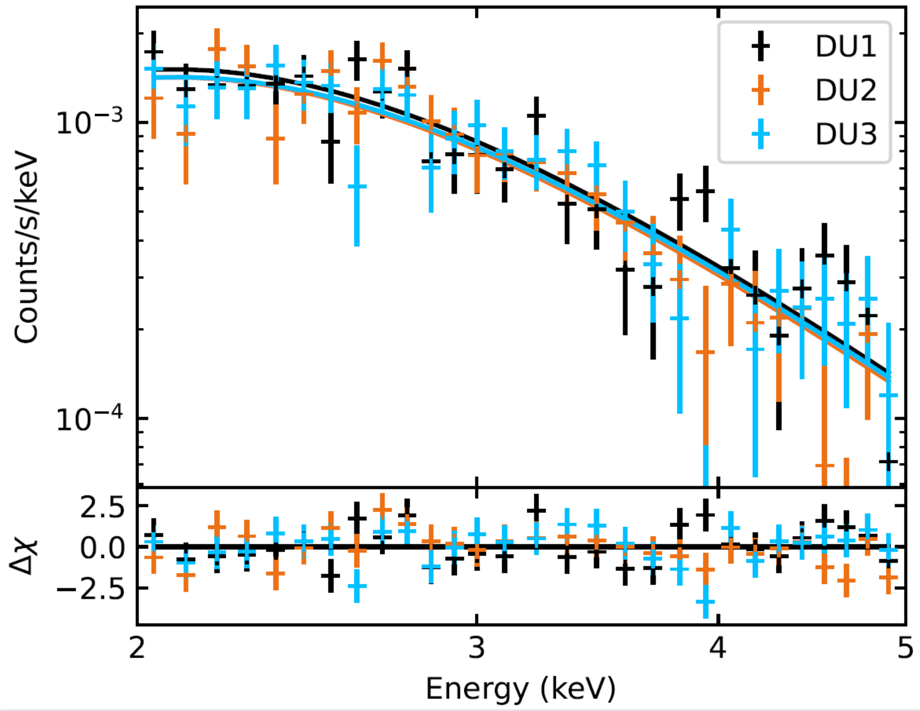
Keep in mind that there are 2 degrees of freedom for each data point, and make sure to account for free parameters if a fit is performed.

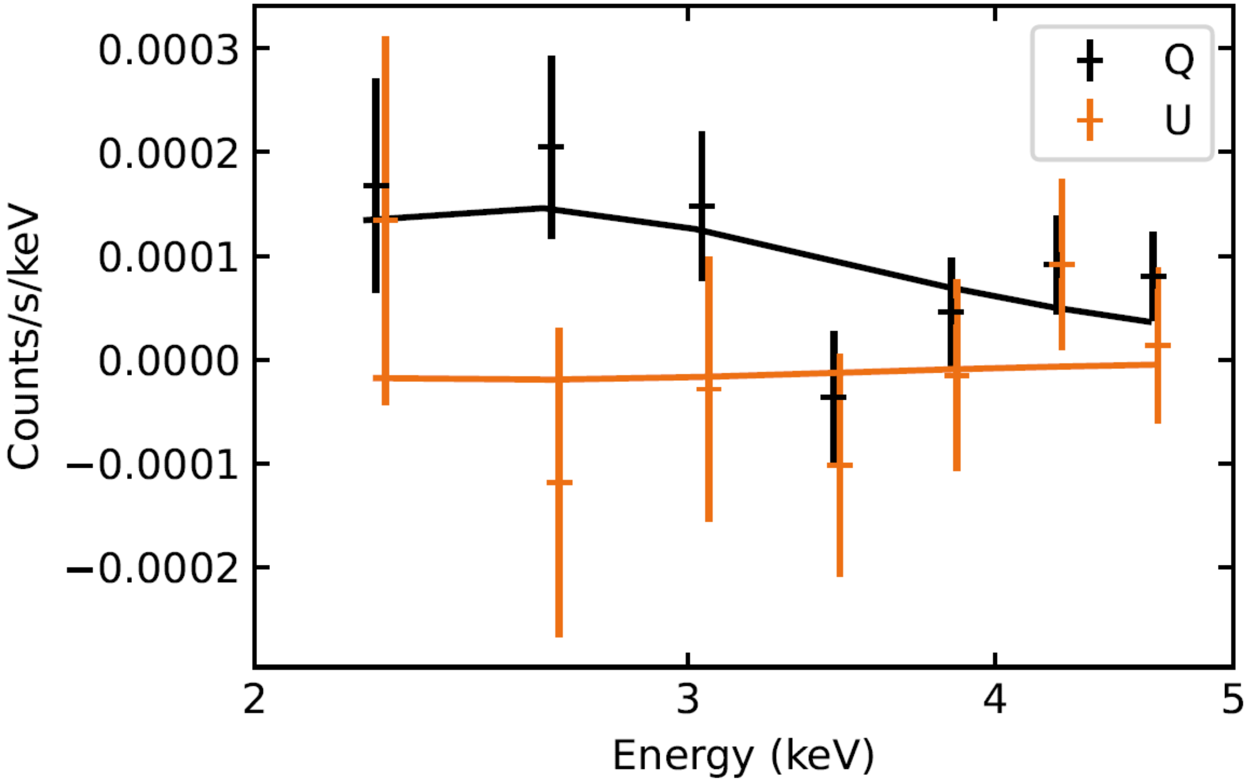


Stokes parameters can be used for a spectro-polarimetric fit.

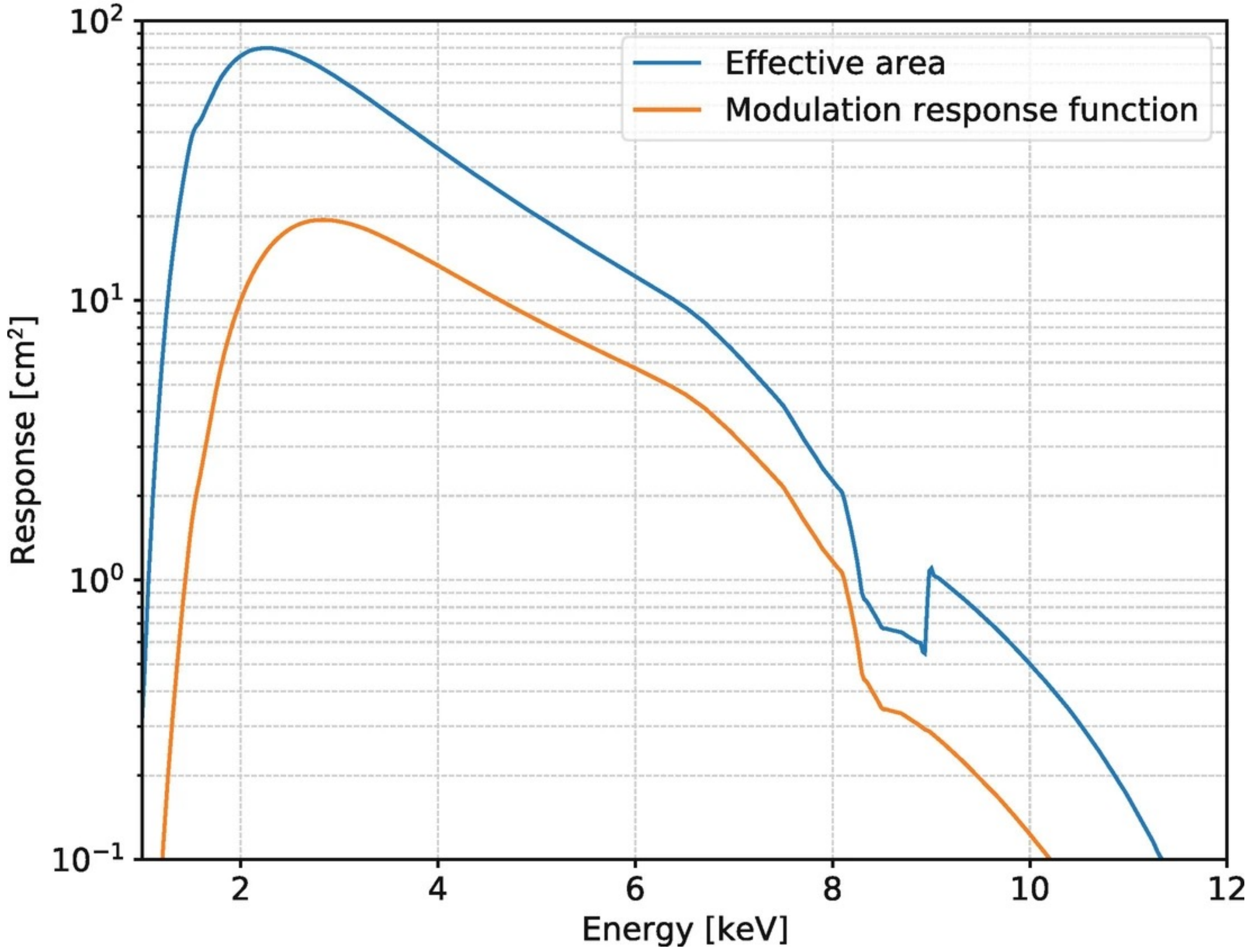
Oftentimes we want to do spectro-polarimetric fits. In these cases, we need to construct spectra for the Stokes I, Q, and U parameters as a function of energy. A typical IXPE spectral fit therefore has 9 data sets fit simultaneous (I/Q/U for DU1/2/3).

The Stokes I spectrum is identical to any other X-ray spectrum you may have fit from another telescope. Note that Q and U spectra, unlike an I spectrum, can be negative.





A spectro-polarimetric fit only differs slightly from a spectral fit.



The only other differences between the Stokes I and Q/U spectra is that the Q/U spectra need a modified ARF file that accounts for the energy-dependent modulation factor and some header keywords set that identify the spectrum as a Stokes Q or U spectrum.

There will be plenty of discussion about how to make these spectra this week.

Note this is the ONLY way to address polarization when there are multiple SED components with different polarization properties.

Introducing weights can improve your polarization sensitivity.

The information gained about polarization is not identical for every event. There are good reasons to provide extra weight to events that are more informative.

$$i_k = w_k$$
$$q_k = 2\, w_k \cos 2\varphi_k$$
$$u_k = 2\, w_k \sin 2\varphi_k$$

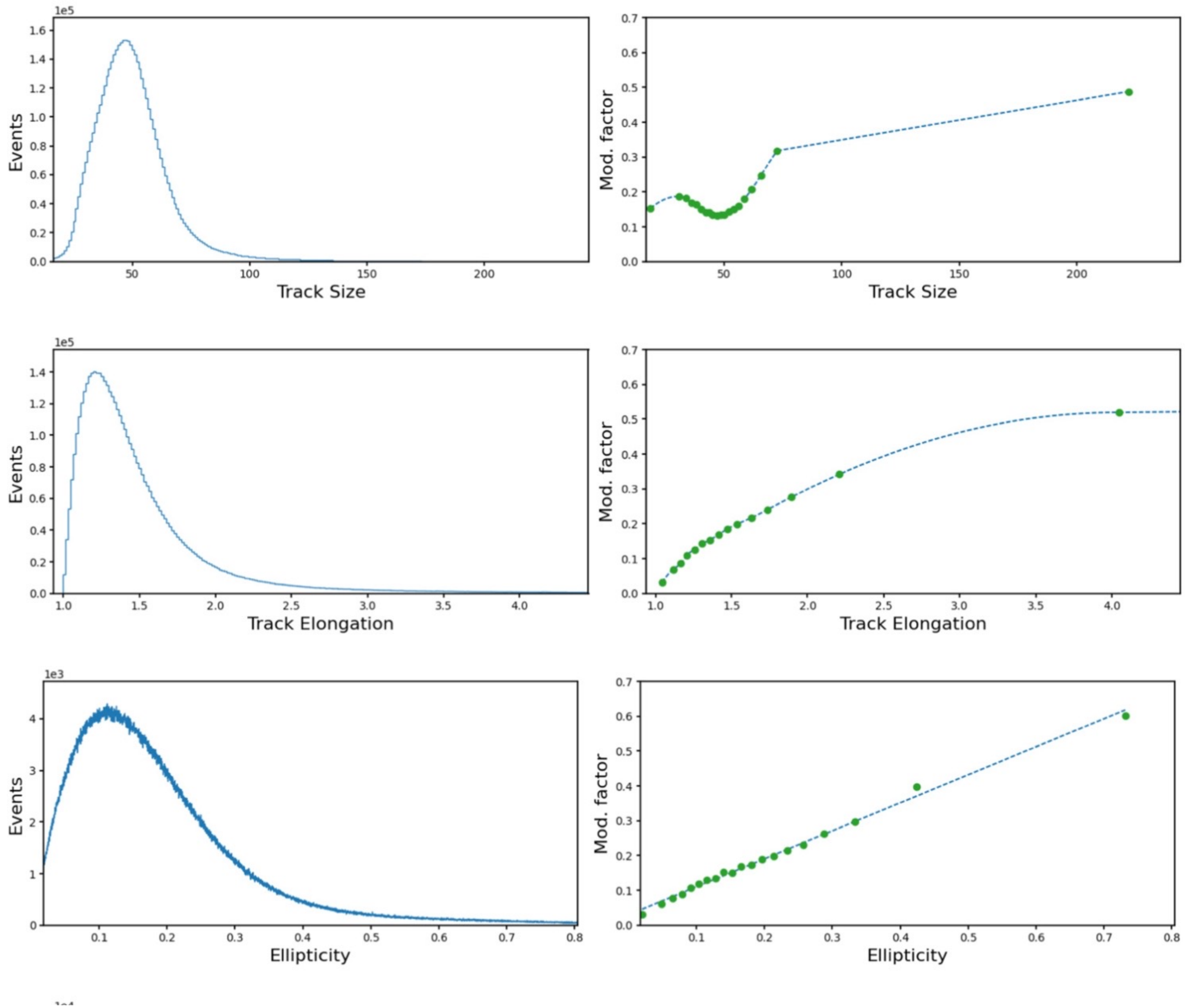
Where the weights are given by w. The polarization degree and angle are determined identically as before using these new versions of the Stokes parameters. Now the effective number of counts is

$$N_{eff} = \sqrt{\frac{\left(\sum_k w_k\right)^2}{\sum_k w_k^2}}$$

What are the proper weights to use?

It can be shown that, for a fixed number of X-ray photons N, the signal-to-noise is maximized when the weight is equal to the event-specific modulation factor. But how do we estimate that?

We estimate it using calibration data with many events and a 100% polarized input beam. These tests have shown that ellipticity (α) is best correlated with modulation factor, and best weight provided by IXPE pipeline is $\alpha^{0.75}$.



When to use weights?

Because the modulation factor and effective area vary so strongly with energy, simulations have shown that some inconsistencies between input values and output results when using weights across a large energy band.

For spectropolarimetric fits and small energy bands this is not an issue.

The typical way the IXPE collaboration has used weights is to NOT use them for simple broad-band measurements or tests but to incorporate them in spectropolarimetric fits.

Concluding Remarks

Polarization measurements typically require vastly more counts than the imaging, spectral, and timing measurements, so please prepare for long exposure times.

Stokes parameters are the straightest path to understanding the statistical distributions associated with polarization detections and measurements.

Weights can help improve your sensitivity to polarization measurements if applied to small energy bands

Both FTOOLS and ixpeobssim analysis tools account for these statistical realities of IXPE data, but as scientists we need to make sure we understand what the software is calculating for us.