Derivation of errors for measurements of unnormalized Stokes Q and U

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1 Definitions

Equation 1 defines the weighted Stokes parameters. Note that $i_k = 1$ for all events and weights (w_k) are included for each event's measured q_k and u_k. The factor 2 are included in the single event q_k and u_k account for the sinusoidal modulation of the azimuth distribution (2). $f(\psi)$ is a function describing the probabilities of measuring different ψ for a given ψ_0, p_0, μ. Note that ψ_k is defined differently for photoelectric and Compton polarimeters. Namely, for Compton polarimeters, a 90° offset is included to account for the fact that photons scatter preferentially perpendicular to their polarization direction. The derivation remains the same for both cases after this offset is included.

$$
\begin{aligned}
q_k &= 2cos(2\psi_k) \\
u_k &= 2sin(2\psi_k) \\
I &= \sum_k i_k w_k \\
Q &= \sum_k w_k q_k \\
U &= \sum_k w_k u_k
\end{aligned}
\tag{1}
$$

$$f(\psi) = \frac{1}{2\pi}(1 + p_0\mu cos(2(\psi - \psi_0))) \tag{2}$$

Here are various identities/integrals that are used:

$$\begin{aligned}
cos(2(x - x_0)) &= cos(2x)cos(2x_0) + sin(2x)sin(2x_0) \\
\int_0^{2\pi} cos(2\psi)d\psi &= 0 \\
\int_0^{2\pi} cos^2(\psi)d\psi &= \pi \\
\int_0^{2\pi} cos^3(\psi)d\psi &= 0 \\
\int_0^{2\pi} cos(2\psi)sin(2\psi) &= 0 \\
\int_0^{2\pi} cos^2(2\psi)sin(2\psi) &= 0 \\
\int_0^{2\pi} sin(2\psi) &= 0
\end{aligned} \tag{3}$$

2 Derive expectation value for single event

First, find the expectation value for a single event's measured $w_k q_k = 2w_k cos(2\psi_k)$.

$$\langle 2w_k cos(2\psi_k)\rangle = \frac{\int_0^{2\pi} 2w_k cos(2\psi)f(\psi)d\psi}{\int_0^{2\pi} f(\psi)d\psi} \tag{4}$$

Since the integral of 2 is normalized to 1, the denominator = 1 and can thus be ignored. Now, plug in $f(\psi)$, and perform the calculation.

$$\begin{aligned}
\langle 2w_k cos(2\psi_k)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} 2w_k cos(2\psi)(1 + p_0\mu cos(2(\psi - \psi_0)))d\psi \\
\langle 2w_k cos(2\psi_k)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} 2w_k cos(2\psi)(1 + p_0\mu(cos(2\psi)cos(2\psi_0) + sin(2\psi)sin(2\psi_0)))d\psi \\
&\text{perform integral and algebra...} \\
\langle 2w_k cos(2\psi_k)\rangle &= w_k p_0 \mu cos(2\psi_0)
\end{aligned} \tag{5}$$

3 Derive variance for single event

Use 5 to calculate the variance for a single event.

$$Var(2w_k cos(2\psi_k)) = \int_0^{2\pi} [2w_k cos(2\psi_k) - \langle 2w_k cos(2\psi_k)\rangle]^2 f(\psi) d\psi \tag{6}$$

Plug in $f(\psi)$ and $\langle 2w_k cos(\psi_k)\rangle$ to find the variance for a single event.

$$Var(2w_k cos(2\psi_k)) = \frac{1}{2\pi}\int_0^{2\pi} [2w_k cos(2\psi) - w_k p_0 \mu cos(2\psi_0)]^2$$
$$*(1 + p_0\mu(cos(2\psi)cos(2\psi_0) + sin(2\psi)sin(2\psi_0)))d\psi$$
$$\text{perform integral and algebra...}$$
$$Var(2w_k cos(2\psi_k)) = w_k^2(2 - p_0^2\mu^2 cos^2(2\psi_0))$$

4 Derive variance for set of events

$$Var(Q) = \sum_k Var(2w_k cos(\psi_k)) \tag{7}$$

Plug in single event variance.

$$\begin{aligned} Var(Q) &= \sum_k w_k^2(2 - p_0^2\mu^2 cos^2(2\psi_0)) \\ &\text{NOTE: } p_0^2\mu^2 cos^2(2\psi_0) = Q^2/I^2 \\ &\text{NOTE: } \sum_k w_k^2 = \sigma_I^2 \\ Var(Q) = \sigma_Q^2 &= \frac{\sigma_I^2}{I^2}(2I^2 - Q^2) \\ &\text{by the same procedure:} \\ Var(U) = \sigma_U^2 &= \frac{\sigma_I^2}{I^2}(2I^2 - U^2) \end{aligned} \tag{8}$$

5 Finalized Errors

$$
\begin{aligned}
\sigma_I^2 &= \sum_{n=1}^{N} w_k^2 \\
\sigma_Q^2 = \frac{\sigma_I^2}{I^2}(2I^2 - Q^2) &\quad if \quad |Q| < I \\
\sigma_Q^2 = \sigma_I^2 &\quad if \quad |Q| > I \\
\sigma_U^2 = \frac{\sigma_I^2}{I^2}(2I^2 - U^2) &\quad if \quad |U| < I \\
\sigma_U^2 = \sigma_I^2 &\quad if \quad |U| > I
\end{aligned}
\tag{9}
$$

The stipulation that $|Q| > I$ and $|U| > I$ are fringe cases. This will only occur in the case of low statistics and high polarization values along specific axes.

These error calculations should proceed through before accounting for the observation time. After all values and errors are calculated, they should simply all be scaled by $1/t_{obs}$ since we assume $\sim$ no error in the time.