imweightadd

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Abstract

Performs a weighted sum of images, with weights optimized for the detection of sources with a give spectrum.

1 Instruments/Modes

Instrument	Mode

2 Use

pipeline processing	yes/no
interactive analysis	yes/no

3 Description

3.1 Introduction

Suppose we have two images of the same part of the sky, taken by the same instrument, in the same energy band and containing the same fraction of background contribution. Clearly, summing these together yields an increase in the sensitivity of detection of point sources. In fact the summed image will be indistinguishable from a single image taken with an exposure duration equal to the sum of the two individual durations.

The situation is otherwise if one of the pair of images contains a much lower ratio of source-to-background than the other. In this case, summing the images can actually lead to a *decrease* in detection sensitivity: in the limiting case it is better just to keep the image with the higher source-to-background and throw the other away.

These two scenarios may be thought of as extreme cases of the more general procedure of forming a weighted sum of the two images. Indeed we may expect that, for the general case where an image I_{sum} is formed as the weighted sum of N contributory images I_i,

$$I_{\mathrm{sum}} = \frac{\sum_{i=1}^{N} w_i I_i}{\sum_{i=1}^{N} w_i},$$

a set of values of the weights w_i exists which maximizes the source-detection sensitivity. It is the task of **imweightadd** to estimate these optimum weights for the given input images and to apply them to calculate I_{sum}.

Before working out an algorithm for calculating the optimum weights it is necessary to say quantitatively what we mean by 'source detection' and 'detection sensitivity.'

3.2 Source detection

Where we have just a single image we assume a probabilistic mode of source detection such that, at a given image pixel, if the probability P that the observed value at that pixel could have arisen through statistical fluctuation of the background at that pixel lies below a certain limiting probability P_{cutoff}, the pixel is considered to contain a source. In XMM practice it is likelihood $L = -\ln P$ rather than P itself which is employed; in this case the detection criterion is for L to exceed the cutoff $L_{\mathrm{cutoff}} = -\ln P_{\mathrm{cutoff}}$.

Matters become more complicated when one has N images taken in different energy bands (or in other differing circumstances). If one has no knowledge of the source spectrum then the best approach is probably to do as **eboxdetect** does, that is to calculate detection likelihood for all the images separately, then add these numbers together. A sum of independent likelihood values like this can be shown to have a null-hypothesis probability distribution P_{sum} approximately given by the formula

$$P_{\mathrm{sum}} = Q\Big(N, \sum_{i=1}^{N} L_i\Big), \qquad (1)$$

where Q is the incomplete gamma function

$$Q(a,x) = \frac{\int_x^\infty dt\ e^{-1}\ t^{a-1}}{\int_0^\infty dt\ e^{-1}\ t^{a-1}}.$$

The final step in the **eboxdetect** approach is to calculate P_{sum} as per equation 1 then test $L_{\mathrm{sum}} = -\ln P_{\mathrm{sum}}$ against the cutoff likelihood as for the single-image case.

The alternative used in the present task is to make a weighted sum of the input images, then perform source detection on the single summed image. For probabilistic detection to work in this scenario we need to find the probability distribution of a weighted sum of Poisson variates. Full discussion of this issue is postponed until section 3.4; suffice it to say here that an approximate expression for this probability distribution has been found. The detection scheme then works like any other: for each pixel, the likelihood that the detected value could have resulted through chance is calculated, and the pixel is designated as a source if the threshhold likelihood is exceeded.

The drawback to the **eboxdetect** approach is that, at image pixels where there is a substantial contribution from a source, the values at this pixel in the different images are no longer independent samples of the background - one expects them all to be higher than the background in general and, if the source spectrum is known, one has additionally some expectation of the ratios at that pixel between the images. It seems likely that the present approach to multi-image detection, which makes use of an *a priori* assumption about the source spectrum, will offer greater sensitivity (to sources which actually have this spectrum) than the **eboxdetect** method. At present there is no analytical proof of this, but empirical trials are planned.

So far I have been talking about source spectra, but it is worth emphasising that the same approach can be used in any situation where the ratio between expectation values of source counts over input images obtained in varying circumstances can be estimated. For instance, suppose one wants to perform source detection upon three images, all made in the same energy band, but each by a different XMM EPIC instrument. The ratio between source counts expected in the three instruments can be estimated from their respective effective area curves. These ratios are, for a narrow enough energy band, insensitive to variations in spectrum from source to source.

3.3 Detection sensitivity

Suppose one has a random variate c with a probability distribution $p(c)$. The nett probability that a given sample of c is due to chance is the integral of p from c to ∞:

$$P(c) = \int_c^\infty dx\ p(x).$$

In the case of a single pixel of a single image, $p(c)$ is the Poisson distribution about the expectation value of the counts due to background, $\langle b \rangle$. As described in section 3.2, we say there is a source present in this pixel if $L = -\ln P$ is larger than a cutoff value L_{cutoff}. In principle it is possible to invert the relationship between detected counts c and likelihood L, to calculate that value of c which would give $L = L_{\text{cutoff}}$. This value of c_{cutoff}, minus the expectation due to background $\langle b \rangle$, is defined here as the detection sensitivity.

Note that this does not mean that a source with an expectation value of counts $\langle s \rangle$ which is greater than $c_{\text{cutoff}} - \langle b \rangle$ will always be detected. There are always statistical fluctuations to consider. The probability that a source with $\langle s \rangle = c_{\text{cutoff}} - \langle b \rangle$ will be detected is the integral from a to infinity of the Poisson distribution with expectation value a equal to $c_{\text{cutoff}} - \langle b \rangle$. This is equal to 0.5 in the limit of large a, but becomes significantly less than 0.5 for $a <$ about 1. Also, the detection cutoff is naturally not sharp: fainter sources have still some non-zero probability of detection, and sources brighter than cutoff have always some non-zero probability of *non*-detection.

Where one is performing source detection in parallel on $N > 1$ images, there are N inputs to the calculation of nett likelihood at any given pixel. In this circumstance it is no longer possible to invert this calculation to obtain a single detection sensitivity, since there may be more than one combination of counts which yield the same nett L. Here a definition of sensitivity only makes sense in connection with a fixed source spectrum, as follows. Suppose that at the pixel in question the expected count values $\langle c_i \rangle$ are made up from background $\langle b_i \rangle$ plus source $\langle s_i \rangle$. (The expectation value of two summed Poisson variates is equal to the sum of the two expectation values.) Suppose also that we know the source spectrum and are thus able to express the source counts as a product between this spectrum and some nett intensity S:

$$\langle s_i \rangle = \langle S \rangle \frac{\langle s_i \rangle}{\sum_i^N \langle s_i \rangle}.$$

Regardless of the precise algorithm employed, likelihood L is some function of the counts c_i, ie

$$L = f(c_i).$$

The detection sensitivity $S_{\rm cutoff}$ can therefore be defined implicitly as follows:

$$L_{\rm cutoff} = f\left(\langle b_i \rangle + S_{\rm cutoff} \frac{\langle s_i \rangle}{\sum_i^N \langle s_i \rangle}\right). \quad (2)$$

3.4 Null-hypothesis probability distribution for a weighted sum of Poissonian variates

The approach followed in the present section is essentially that of Fay and Feuer [1]. See also Stewart [**?**]

Let X be a random deviate which follows a Poissonian probability distribution about the expectation value $a = \langle X \rangle$. Although the Poisson distribution itself is only defined for integer X, one can find the following continuous 'envelope function' to the Poisson values:

$$e(X; a) = \frac{a^X \exp(-a)}{\Gamma(X+1)}, \quad (3)$$

where Γ is the gamma function

$$\Gamma(a) = \int_0^\infty dt\ e^{-1}\ t^{a-1}.$$

Note that both the expectation value $\langle X^2 \rangle$ and variance $\langle X \rangle - \langle X \rangle^2$ of the function e are identical to those of the corresponding discrete Poisson distribution, namely both equal to a. Now, given N random deviates X_i, each of which follows a distinct Poisson distribution about its average a_i, let us form the weighted sum

$$Y = \sum_{i=1}^N w_i X_i.$$

The expectation value μ of Y is

$$\mu = \langle Y \rangle = \sum_{i=1}^N w_i \langle X_i \rangle = \sum_{i=1}^N w_i a_i;$$

the variance v can in similar fashion be shown to equal

$$v = \sum_{i=1}^N w_i^2 a_i.$$

The probability function $p(Y)$ only has values where all the X_i are integer and generally speaking may be expected to be a messy-looking and intractable function. However, recall that for purposes of source detection we are not interested in p but in the integral P of p from a particular sample y of Y to infinity.

P is stepwise continuous, the steps becoming smaller and denser as y increases. The envelope function with the same expectation value and variance as Y is given by

$$e(Y;\mu,v) = \frac{(k\mu)^{(kY)} exp(-k\mu)}{\Gamma(kY+1)}, \tag{4}$$

where

$$k = \mu/v.$$

In plain English, what comparison of equations 3 and 4 suggests is that a weighted sum of Poisson variates behaves approximately like a single Poisson distribution with the same average and variance. Therefore since, for a single Poisson variate X, the probability P for X to be greater than some sampled value x is given by

$$P(X \geq x; a) = 1 - Q(x, a)$$

(where Q is the incomplete gamma function we met with in section 3.2) we postulate that the equivalent expression for Y is approximately given as follows:

$$P(Y \geq y; \mu, v) \sim 1 - Q(ky, k\mu). \tag{5}$$

Equation 5 is used both in the present program and in **boxdetect** to estimate the null-hypothesis probability distribution of the weighted sum of Poissonian images.

3.5 Estimation of the optimal weights

In order to estimate the optimal weights, **imweightadd** needs two lists: a list of background maps and a list of source relative expectation values σ_i.

The first job is to reduce each background map to a single representative value β_i of background counts per pixel. This is done by making a histogram of all the non-zero values in each map and selecting the value which falls nearest the 90% mark on the histogram. The rationale behind this is as follows. Source detection is likely to be most sensitive near the centre of the image; this is also the place where one would expect the maximum to be in the background values. Hence it makes sense to choose a value which is nearer to the maximum value than to the minimum. However there is also the possibility of local increases in background due to out-of-time events or such like. Because of this possibility it was thought undesirable to pluck the background value right from the top of the tree so to speak: hence the 90% figure was arrived at as a compromise.

The next thing is to normalize the σs to 1. Naturally at least one of them must be non-zero.

imweightadd then performs a minimisation via a simplex algorithm. The quantity to be minimized is the detection sensitivity as defined in section 3.3. The analogue of equation 2 in the present case is

$$L_{\text{cutoff}} = -\ln[1 - Q(ky_{\text{cutoff}}, k\mu)] \tag{6}$$

where

$$y_{\rm cutoff} = \sum_{i=1}^{N} w_i(\beta_i + S_{\rm cutoff}\sigma_i),$$

$$\mu = \sum_{i=1}^{N} w_i\beta_i,$$

$$v = \sum_{i=1}^{N} w_i^2\beta_i,$$

and

$$k = \mu/v$$

as before. At each step, equation 6 is inverted numerically via a Ridders-method algorithm to yield the sensitivity $S_{\rm cutoff}$. The minimization procedure therefore arrives at the set of weights which yield the minimum value (within convergence limits) of $S_{\rm cutoff}$. These weights are then applied to generate weighted sums of the input images and also the input background maps and exposure maps.

4 Parameters

This section documents the parameters recognized by this task (if any).

Parameter	Mand	Type	Default	Constraints

imagesets	yes	dataset list		

List of N input images to be summed. They must all have the same pixel dimensions.

outimageset	no	dataset	outimage.ds	

The output image, equal to $\sum_{i=1}^{N} w_i$ `imagesets`.

tempset	no	dataset	tempimage.ds	

Name of a temporary image dataset (for pipeline use).

calculateweights	no	bool	no	

If this parameter = 'no', the input images are simply summed, and neither background maps or exposure maps are required inputs. Contrariwise for 'yes'.

bkgmapsets	yes	dataset list		

List of background maps to be summed. They must all have the same pixel dimensions as the `imagesets`, and there must be the same number of members in each list. Each background map should correspond with the image at the same place in the respective list. This parameter is only read if `calculateweights`='yes'.

outbkgmapset	no	dataset	outbkgmap.ds	

The output background map, equal to $\sum_{i=1}^{N} w_i$ `bkgmapsets`. This parameter is only read if `calculateweights`=‘yes’.

expmapsets	yes	dataset list		

List of exposure maps to be summed. They must all have the same pixel dimensions as the `imagesets`, and there must be the same number of members in each list. Each exposure map should correspond with the image at the same place in the respective list. This parameter is only read if `calculateweights`=‘yes’.

outexpmapset	no	dataset	outexpmap.ds	

The output exposure map, equal to $\sum_{i=1}^{N} w_i$ `expmapsets`. This parameter is only read if `calculateweights`=‘yes’.

withrelsrcrates	no	bool	no	

Whether to read the relative source-count expectation values (‘source spectrum’) from parameter `relsrcrates`. If `withrelsrcrates`=‘no’, these relative rates are all set to 1. This parameter is only read if `calculateweights`=‘yes’.

relsrcrates	yes	real list	1.0	0 <`relsrcrates`

This parameter is read if `calculateweights`=‘yes’ and `withrelsrcrates`=‘yes’.

likemin	no	real	10.0	0 <`likemin`

The cutoff value of likelihood.

5 Errors

This section documents warnings and errors generated by this task (if any). Note that warnings and errors can also be generated in the SAS infrastructure libraries, in which case they would not be documented here. Refer to the index of all errors and warnings available in the HTML version of the SAS documentation.

label *(error)*
explanation

label *(warning)*

corrective action: ********

6 Input Files

1. ********

7 Output Files

1. ********

8 Algorithm

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9 Comments

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References

[1] M. P. Fay and E. J. Feuer. Confidence intervals for directly standardized rates: a method based on the gamma distribution. *Statistics in Medicine*, 16:791–801, 1997.